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Weighted sampling and reconstruction in weighted reproducing kernel spaces ☆

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ABSTRACT

In this paper, we discuss sampling and reconstruction of signals in the weighted reproducing kernel space associated with an idempotent integral operator. We show that any signal in that space can be stably reconstructed from its weighted samples taken on a relatively-separated set with sufficiently small gap. We also develop an iterative reconstruction algorithm for the reconstruction of a signal from its weighted samples taken on a relatively-separated set with sufficiently small gap.

1. Introduction

The reconstruction of a signal f on the d-dimensional Euclidean space $\mathbb{R}^d$ from its samples $\{f(x) \mid x \in X\}$ taken on a countable set X is a common task in many applications in signal (or image) processing. It is also a fundamental problem in sampling theory.

The above problem is obviously an ill-posed inverse problem in general, but is solvable with additional constraint on the signal f. For instance, in the famous Shannon's sampling theorem, signals to be reconstructed are assumed to be band-limited with finite energy [13]. In that case, a signal f can be perfectly reconstructed from its uniform samples $f(n/T)$, $n \in \mathbb{Z}$, with sampling rate $1/T$ greater than twice the bandwidth B.

In many engineering applications, such as MRI imaging, signals and images are not band-limited. The shift-invariant space model developed in 1990s is successful for many engineering problems, where the signal f to be reconstructed is assumed to live in a shift-invariant space. It has been shown to be suitable and realistic, especially for taking into account of realistic environment, for modeling signals with smooth spectrum, or for numerical implementation [2,6,7,16,18–20,23].

In engineering problems such as ultra wideband communication and compressing sensing, the signal (or the image) does not have shift-invariant structure. Most of those signals have finite rate of innovation, see [5,8,14,15,21].

Let $L^p := L^p(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$, be the space of all p-integrable functions on $\mathbb{R}^d$ with the standard norm $\|\cdot\|_p$. A *reproducing kernel subspace of* L^p [4] is a closed subspace V of L^p such that for any $x \in \mathbb{R}^d$ there exists a positive constant C_x such that

$$\left|f(x)\right| \leqslant C_x \|f\|_p \quad \text{for all } f \in V. \tag{1.1}$$

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Nashed and Sun recently introduced a reproducing kernel space model and studied the sampling and reconstruction of signals in that reproducing kernel subspace of L^p from its samples [12]. The space of p-integrable non-uniform splines and the shift-invariant spaces generated by finite many localized functions are their model examples of such reproducing kernel space. In this paper, we will introduce a *weighted reproducing kernel subspace*. The weighted reproducing kernel subspace model accommodate the reproducing kernel subspace model. In practice, the assumption that the ideal sampled values $\{f(\gamma)\}$ can be measured exactly is not realistic. A better assumption is that the sampled data is of the form $\langle f, \psi_\gamma\rangle = \int_{\mathbb{R}^d} f(x)\psi_\gamma(x)\,dx$, where ψ_γ to be known as the weighted sampling functional [2,3,9,14,18,19,22,23]. The conclusions are only established in [12] for the ideal sampling and trivial weight case. In this paper, we will work on the weighted sampling and reconstruction problem for signals in a reproducing kernel subspace of the weighted L^p space.

The paper is organized as follows. In Section 2, we recall some concepts such as weighted reproducing kernel spaces for signals to live in and relatively-separated sets for sampling, and establish some preliminary results. In Section 3, we show that signals, in a reproducing kernel subspace associated with an integral idempotent operator with kernels satisfying certain decay condition at infinity and regularity condition, can be stably reconstructed from its average samples taken on a relatively-separated set with small gap (Theorem 3.1). In Section 4, we introduce an iterative algorithm to reconstruct a signal in the above reproducing kernel subspace from its weighted samples taken on a relatively-separated set with small gap (Theorem 4.1).

2. Preliminaries

In this section, we recall the concepts of weights, reproducing kernel subspaces of a weighted L^p space, idempotent operators, reproducing kernel subspaces associated with an idempotent operator, and relatively-separated subsets of $\mathbb{R}^d$. We also establish some properties about weights and reproducing kernel spaces.

2.1. Weights

A *weight* w in this paper is a positive function w on $\mathbb{R}^d$, i.e., $w(x) > 0$ for all $x \in \mathbb{R}^d$. A *submultiplicative weight* w is a weight that satisfies

$$w(x+y) \leqslant w(x)w(y) \quad \text{for all } x, y \in \mathbb{R}^d. \tag{2.1}$$

The polynomial weights $(1+|x|)^\alpha$ and the (sub)exponential weights $\exp(\alpha|x|^\delta)$, where $0 \leqslant \alpha \in \mathbb{R}$ and $\delta \in [0,1]$, are our model examples of submultiplicative weights w.

A weight v is said to be *moderate with respect to the weight* w, or simply w*-moderate*, if

$$v(x+y) \leqslant w(x)v(y) \quad \text{for all } x, y \in \mathbb{R}^d. \tag{2.2}$$

For a w-moderate weight v,

$$\big(w(-x)\big)^{-1} v(y) \leqslant v(x+y) \leqslant w(x)v(y) \tag{2.3}$$

and

$$\big(w(y-x)\big)^{-1}\big(v(x)\big)^{-1} \leqslant \big(v(y)\big)^{-1} \leqslant w(x-y)\big(v(x)\big)^{-1} \tag{2.4}$$

hold for all $x, y \in \mathbb{R}^d$.

2.2. Weighted L^p and ℓ^p spaces

Given $p \in [1,\infty]$ and a weight v, the weighted L^p space $L^p_v := L^p_v(\mathbb{R}^d)$ contains all measurable functions f with $fv \in L^p$, the space of all p-measurable functions on $\mathbb{R}^d$. Equipped with the norm $\|f\|_{p,v} = \|vf\|_p$ for $f \in L^p_v$, the weighted space L^p_v becomes a Banach space. For integral operators on the weighted space L^p_v, we have the following result.

Proposition 2.1. *Let $1 \leqslant p \leqslant \infty$, w be a weight, v be a w-moderate weight, and K be a measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ that satisfies the following decay condition*

$$\|K\|_{\mathcal{W}_{1,w}} := \Big(\sup_{x\in\mathbb{R}^d} \big\|K(x, x-\cdot)\big\|_{1,w}\Big)^{1-1/p} \Big(\sup_{y\in\mathbb{R}^d} \big\|K(\cdot+y, y)\big\|_{1,w}\Big)^{1/p} < \infty. \tag{2.5}$$

Then the linear operator T defined by

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y) f(y)\,dy, \quad f \in L^p_v \tag{2.6}$$

is a bounded operator on L_v^p. Moreover

$$\|Tf\|_{p,v} \leqslant \|K\|_{\mathcal{W}_{1,w}} \|f\|_{p,v} \quad \textit{for all } f \in L_v^p. \tag{2.7}$$

Let $\ell^p := \ell^p(\mathbb{Z}^d)$ be the space of all p-summable sequences with norm $\|\cdot\|_{\ell^p}$. Given a positive sequence $v = (v(j))_{j\in\mathbb{Z}^d}$ on $\mathbb{Z}^d$, the weight space $\ell_v^p := \ell_v^p(\mathbb{Z}^d)$ contains all sequences $c = (c(j))_{j\in\mathbb{Z}^d}$ on $\mathbb{Z}^d$ with $\|c\|_{\ell_v^p} := \|cv\|_{\ell^p} < \infty$, where $cv := (c(j)v(j))_{j\in\mathbb{Z}^d}$.

2.3. Reproducing kernel subspaces of L_v^p

A *reproducing kernel subspace of L_v^p*, or a *weighted reproducing kernel space*, is a closed subspace V_v of L_v^p such that for any $x \in \mathbb{R}^d$ there exists a positive constant C_x such that

$$\left|f(x)\right| \leqslant C_x \|f\|_{p,v} \quad \text{for all } f \in V_v. \tag{2.8}$$

An *idempotent* operator T [12] on L_v^p is a bounded linear operator on L_v^p that satisfies

$$T^2 f = Tf \quad \text{for all } f \in L_v^p. \tag{2.9}$$

Denote by V_v the *range space* of the idempotent operator T on L_v^p, i.e.,

$$V_v := \left\{Tf \;\middle|\; f \in L_v^p\right\}. \tag{2.10}$$

We say that the range space V_v of the idempotent operator T on L_v^p is a *reproducing kernel space associated with the idempotent operator T on L_v^p*, or a *weighted reproducing kernel space associated with the idempotent operator T*, if it is a reproducing kernel subspace of L_v^p. For any f in the range space,

$$Tf = f, \tag{2.11}$$

and

$$\left|f(x)\right| \leqslant \int_{\mathbb{R}^d} \left|K(x,y)\right|\left|f(y)\right| dy \leqslant \left\|K(x,\cdot)\big(v(\cdot)\big)^{-1}\right\|_{p/(p-1)} \|f\|_{p,v}.$$

Therefore we have the following reproducing kernel space property for the range space associated with an idempotent operator.

Proposition 2.2. *Let $1 \leqslant p \leqslant \infty$, w be a weight, v be a w-moderate weight, K be a measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ that satisfies* (2.5) *and*

$$\left\|K(x,\cdot)\big(v(\cdot)\big)^{-1}\right\|_{p/(p-1)} < \infty \quad \textit{for all } x \in \mathbb{R}^d, \tag{2.12}$$

and define the integral operator T with kernel K as in (2.6)*. Assume that T is an idempotent operator on L_v^p. Then the range space V_v of the idempotent operator T on L_v^p is a reproducing kernel subspace of L_v^p.*

Define the *modulus of continuity* $\omega_\delta(f)$ of a function f on $\mathbb{R}^d$ by

$$\omega_\delta(f)(x) = \sup_{x' \in [-\delta,\delta]^d} \left|f\big(x+x'\big) - f(x)\right|, \tag{2.13}$$

and the modulus of continuity $\omega_\delta(K)$ of a kernel function K on $\mathbb{R}^d \times \mathbb{R}^d$ by

$$\omega_\delta(K)(x,y) = \sup_{x',y' \in [-\delta,\delta]^d} \left|K\big(x'+x, y'+y\big) - K(x,y)\right|. \tag{2.14}$$

Proposition 2.3. *Let $1 \leqslant p \leqslant \infty$, w be a continuous submultiplicative weight, and v be a continuous w-moderate weight. If a measurable function K on $\mathbb{R}^d \times \mathbb{R}^d$ satisfies*

$$\max\left(\left\|K(x,x-\cdot)\right\|_{1,w}, \left\|\omega_\delta(K)(x,x-\cdot)\right\|_{1,w}\right) < \infty \quad \textit{for all } x \in \mathbb{R}^d$$

where $\delta > 0$, then (2.12) *holds.*

Proof. By the definition of the modulus of continuity for the kernel K,

$$\begin{cases} \left|K(x,y)\right| \leqslant \left|K(x,\delta k)\right| + \omega_{\delta/2}(K)(x,\delta k), \\ \left|K(x,\delta k)\right| \leqslant \delta^{-d} \displaystyle\int_{[-\delta/2,\delta/2]^d} \left(\left|K(x,z)\right| dz + \left|\omega_{\delta/2}(K)(x,z)\right|\right) dz, \\ \omega_{\delta/2}(K)(x,\delta k) \leqslant \delta^{-d} \displaystyle\int_{[-\delta/2,\delta/2]^d} \left(\omega_{\delta/2}(K)(x,z) + \omega_{\delta}(K)(x,z)\right) dz, \end{cases} \tag{2.15}$$

where $y \in \delta k + [-\delta/2, \delta/2]^d$, $k \in \mathbb{Z}^d$ and $x \in \mathbb{R}^d$. By (2.1) and (2.4), we obtain

$$\begin{cases} \left(v(y)\right)^{-1} \leqslant w(-\delta k)\left(v(y-\delta k)\right)^{-1} \leqslant C_1 w(-\delta k), \\ w(-\delta k) \leqslant w(y-\delta k) w(x-y) w(-x) \leqslant C_2 w(x-y) w(-x), \end{cases} \tag{2.16}$$

where $C_1 = \sup_{z\in[-\delta/2,\delta/2]^d} (v(z))^{-1} < \infty$ and $C_2 = \sup_{z\in[-\delta/2,\delta/2]^d} w(z) < \infty$ by the continuity of the weights v and w. Therefore

$$\begin{aligned} &\left\|K(x,\cdot)\left(v(\cdot)\right)^{-1}\right\|_{p/(p-1)} \\ &\quad\leqslant C_1 \delta^{d(p-1)/p}\left(\left\|\left(\left|K(x,\delta k)\right| w(-\delta k)\right)_{k\in\mathbb{Z}^d}\right\|_{\ell^p} + \left\|\left(\omega_{\delta/2}(K)(x,\delta k) w(-\delta k)\right)_{k\in\mathbb{Z}^d}\right\|_{\ell^p}\right) \\ &\quad\leqslant C_1 \delta^{d(p-1)/p}\left(\left\|\left(\left|K(x,\delta k)\right| w(\delta k)\right)_{k\in\mathbb{Z}^d}\right\|_{\ell^1} + \left\|\left(\omega_{\delta/2}(K)(x,\delta k) w(\delta k)\right)_{k\in\mathbb{Z}^d}\right\|_{\ell^1}\right) \\ &\quad\leqslant C_1 C_2 w(-x) \delta^{d/p}\left(\left\|K(x,\cdot) w(x-\cdot)\right\|_1 + 2\left\|\omega_{\delta/2}(K)(x,\cdot) w(x-\cdot)\right\|_1 + \left\|\omega_{\delta}(K)(x,\cdot) w(x-\cdot)\right\|_1\right) \\ &\quad\leqslant C_1 C_2 \delta^{d/p} w(-x)\left(\left\|K(x,x-\cdot)\right\|_{1,w} + 3\left\|\omega_{\delta}(K)(x,x-\cdot)\right\|_{1,w}\right) < \infty, \end{aligned} \tag{2.17}$$

and (2.12) follows. □

In this paper, we **always** assume that signals to be sampled and reconstructed live in a weighted reproducing kernel subspace V_v associated with an integral idempotent operator T in (2.6) whose kernel K satisfies the decay condition (2.5) and the following regularity condition

$$\left\|\omega_\delta(K)\right\|_{\mathcal{W}_{1,w}} := \left(\sup_{x\in\mathbb{R}^d}\left\|\omega_\delta(K)(x,x-\cdot)\right\|_{1,w}\right)^{1-1/p}\left(\sup_{y\in\mathbb{R}^d}\left\|\omega_\delta(K)(\cdot+y,y)\right\|_{1,w}\right)^{1/p} < \infty \tag{2.18}$$

for some $\delta > 0$. In this case, it follows from Propositions 2.2 and 2.3 that the integral operator T is a bounded operator on L^p_v and that the range space V_v associated with the operator T is a reproducing kernel subspace of L^p_v.

In the following, we present one example of a reproducing kernel space associated with an idempotent integral operator on L^p_v.

Example 2.1. (See [2].) Let v be w-moderate,

$$\phi \in W\left(L^1_w\right) := \left\{ f \;\middle|\; \|f\|_{W(L^1_w)} = \sum_{k\in\mathbb{Z}^d} \operatorname{esssup}\left\{\left|f(x+k)\right| w(k);\ x \in [0,1]^d\right\} < \infty \right\}$$

and $\tilde{\phi} \in W(L^1_w)$ be the dual of ϕ such that

$$\left\langle\tilde{\phi}(\cdot-j), \phi(\cdot-i)\right\rangle = \delta_{i,j} \quad \text{for all } i, j \in \mathbb{Z}^d,$$

where $\delta_{i,j}$ stands for the Kronecker symbol. Define the kernel function K

$$K(x,y) = \sum_{k\in\mathbb{Z}^d} \overline{\phi(x-k)}\tilde{\phi}(y-k).$$

Then the operator T with the kernel K is an idempotent operator, that is, $Tf = f$. If $\omega_\delta(\phi)$ and $\omega_\delta(\tilde{\phi})$ satisfy that $\lim_{\delta\to 0}\|\omega_\delta(\phi)\|_{W(L^1_w)} = \lim_{\delta\to 0}\|\omega_\delta(\tilde{\phi})\|_{W(L^1_w)} = 0$, then one may verify that the kernel function $K(x,y)$ satisfies (2.5) and (2.18). In this case,

$$V := \left\{ \sum_{k\in\mathbb{Z}^d} c(k)\phi(x-k):\ c \in \ell^2_v \right\}$$

is the range space of operator T on L^2_v and hence a reproducing kernel subspace of L^2_v.

2.4. Sampling sets

Denote by χ_E the characteristic function on a measurable set E. A discrete subset Γ of $\mathbb{R}^d$ is said to be *relatively-separated* if

$$B_\Lambda(\delta) := \sup_{x\in\mathbb{R}^d} \sum_{\gamma\in\Gamma} \chi_{\gamma+[-\delta/2,\delta/2]^d}(x) < \infty \tag{2.19}$$

for some $\delta > 0$. A positive number δ is said to be a *gap* of a relatively-separated subset Γ of $\mathbb{R}^d$ if

$$A_\tau(\delta) := \inf_{x\in\mathbb{R}^d} \sum_{\gamma\in\Gamma} \chi_{\gamma+[-\delta/2,\delta/2]^d}(x) \geqslant 1. \tag{2.20}$$

For a relatively-separated subset Γ of $\mathbb{R}^d$ having position gap, we define the smallest positive number δ with $A_\Gamma(\delta) \geqslant 1$ as its *maximal gap*.

In this paper, we **always** assume that the samples $y_\gamma := f(\gamma)$, $\gamma \in \Gamma$, of the signal to be reconstructed are taken on a relatively-separated subset Γ of $\mathbb{R}^d$ with positive gap. In fact, we generalize the above ideal sampling procedure by considering weighted samples,

$$y_\gamma := \langle f, \psi_\gamma\rangle = \int_{\mathbb{R}^d} f(x)\psi_\gamma(x)\,dx, \quad \gamma \in \Gamma, \tag{2.21}$$

which is a weighted average of sampling values of the signal f in the small neighborhood of γ. The weighted sampling functionals ψ_γ in the above weighted sampling reflects the characteristic of the sampling device located at the sampling location $\gamma \in \Gamma$. The sampling procedure (2.21) is also known as *average sampling* [3,18], *local average sampling* [19] and *generalized sampling* [9]. We describe the sampling process (2.21) as weighted sampling, as in the paper the sampling functional ψ_γ is supported in a *small* neighborhood of the location γ for each $\gamma \in \Gamma$. The reader may refer to [2,3,9,14,18,19,22,23] for weighted sampling of a band-limited signal, a signal in a shift-invariant space, or a signal with finite rate of innovation.

3. Weighted sampling

In this section, we consider the problem when a signal f in a reproducing kernel subspace V_ν of the weighted L^p space can be uniquely determined and stably reconstructed from its weighted samples $\langle f, \psi_\gamma\rangle$, $\gamma \in \Gamma$, taken a relatively-separated subset Γ of $\mathbb{R}^d$ with sufficiently small gap. In fact, we are going to establish the following stability inequality:

$$A\|f\|_{p,\nu} \leqslant \big\|\big(\langle f, \psi_\gamma\rangle\big)_{\gamma\in\Gamma}\big\|_{\ell^p_\nu} \leqslant B\|f\|_{p,\nu} \quad \text{for all } f \in V_\nu, \tag{3.1}$$

where A and B are positive constants.

Theorem 3.1. *Let $1 \leqslant p \leqslant \infty$, $0 < \delta_0$, $a < \infty$, w be a continuous submultiplicative weight, ν be a continuous w-moderate weight, K be a measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying the decay condition (2.5) and the regularity condition (2.18) for $\delta = (\delta_0 + a)/2$, T be the integral idempotent operator in (2.6) with kernel K, V_ν be the reproducing kernel subspace of L^p_ν associated with the operator T, Γ be a relatively-separated subset of $\mathbb{R}^d$ with δ_0 being its gap, and $\Psi = \{\psi_\gamma\}_{\gamma\in\Gamma}$ be a family of weighted sampling functionals with the property that $M := \sup_{\gamma\in\Gamma}\|\psi\|_1 < \infty$, $\int_{\mathbb{R}^d}\psi_\gamma(x)\,dx = 1$ for each $\gamma \in \Gamma$, and ψ_γ is supported in $\gamma + [-a/2, a/2]^d$ for each $\gamma \in \Gamma$. Suppose that*

$$r_0 := M\Big(\sup_{x\in\mathbb{R}^d}\big\|\omega_{(\delta_0+a)/2}(K)(x, x-\cdot)\big\|_{1,w}\Big)^{1-1/p}\Big(\sup_{y\in\mathbb{R}^d}\big\|\omega_{(\delta_0+a)/2}(K)(y+\cdot, y)\big\|_{1,w}\Big)^{1/p} < 1. \tag{3.2}$$

Then any signal $g \in V_\nu$ can be reconstructed from its average samples $\langle g, \psi_\gamma\rangle$, $\gamma \in \Gamma$, in a stable way. Moreover,

$$\begin{aligned}
&(1 - r_0)\big(\delta_0^{-d}A_\Gamma(\delta_0)\big)^{1/p}\Big(\sup_{z\in[-\delta_0/2,\delta_0/2]} w(z)\Big)^{-1}\|g\|_{p,\nu} \\
&\quad\leqslant \big\|\big(\langle g, \psi_\gamma\rangle\big)_{\gamma\in\Gamma}\big\|_{\ell^p_\nu} \\
&\quad\leqslant (M + r_0)\big(\delta_0^{-d}B_\Gamma(\delta_0)\big)^{1/p}\Big(\sup_{z\in[-\delta_0/2,\delta_0/2]^d} w(z)\Big)\|g\|_{p,\nu}.
\end{aligned} \tag{3.3}$$

Remark 3.1. The conclusions in Theorem 3.1 are established in [12] for the ideal sampling and trivial weight case, i.e., $\langle g, \psi_\gamma\rangle = g(\gamma)$ for all $\gamma \in \Gamma$ and $\nu = w \equiv 1$. The stable reconstruction conclusion established in Theorem 3.1 is usually known as samplability property for signals in the weighted reproducing kernel subspace V_ν. About samplability, the reader may refer to [10,13,20] for band-limited signals, [2,3,14,22] for signals in a shift-invariant space, [17,22] for signals in a spline subspace and [11,12] for signals in a reproducing kernel space.

Now we prove Theorem 3.1.

Proof of Theorem 3.1. Clearly it suffices to establish (3.3).

Given a relatively-separated subset Γ with positive gap δ_0, $U=\{u_\gamma\}_{\gamma\in\Gamma}$ form a *bounded uniform partition of unity* (*BUPU*) associated with the covering $\{\gamma+[-\delta_0/2,\delta_0/2]^d\}_{\gamma\in\Gamma}$, i.e.,

$$\begin{cases} 0\leqslant u_\gamma\leqslant 1 \quad \text{for each } \gamma\in\Gamma, \\ u_\gamma \text{ is supported in } \gamma+[-\delta_0/2,\delta_0/2]^d \quad \text{for each } \gamma\in\Gamma, \\ \sum_{\gamma\in\Gamma} u_\gamma(x)=1 \quad \text{for all } x\in\mathbb{R}^d. \end{cases} \tag{3.4}$$

Define the approximation operator Q from L^p_v to L^p_v by

$$Qf(x)=\sum_{\gamma\in\Gamma}\langle Tf,\psi_\gamma\rangle u_\gamma(x), \quad f\in L^p_v. \tag{3.5}$$

Then for any $f\in L^p_v$,

$$\begin{aligned} \|Tf-Qf\|_{p,v} &= \left\| \int_{\mathbb{R}^d} \left(\sum_{\gamma\in\Gamma} \left(\int_{\mathbb{R}^d} \big(K(\cdot,z)-K(y,z)\big)\phi_\gamma(y)\,dy \right) u_\gamma(\cdot) \right) f(z)\,dz \right\|_{p,v} \\ &\leqslant \left\| \int_{\mathbb{R}^d} \left(\sum_{\gamma\in\Gamma} \left(\int_{\mathbb{R}^d} \omega_{(\delta_0+a)/2}(K)(\cdot,z)\big|\phi_\gamma(y)\big|\,dy \right) u_\gamma(\cdot) \right) \big|f(z)\big|\,dz \right\|_{p,v} \\ &\leqslant M \left\| \int_{\mathbb{R}^d} \omega_{(\delta_0+a)/2}(K)(\cdot,z)\big|f(z)\big|\,dz \right\|_{p,v} \\ &\leqslant M \Big(\sup_{x\in\mathbb{R}^d} \big\|\omega_{(\delta_0+a)/2}(K)(x,x-\cdot)\big\|_{1,w} \Big)^{1-1/p} \\ &\quad \times \Big(\sup_{y\in\mathbb{R}^d} \big\|\omega_{(\delta_0+a)/2}(K)(y+\cdot,y)\big\|_{1,w} \Big)^{1/p} \|f\|_{p,v} =: r_0\|f\|_{p,v}, \end{aligned} \tag{3.6}$$

where the first and second inequalities follow from (3.4) and the definition of the modulus of continuity, and the third inequality is true by Proposition 2.1. Combining (2.11), (3.2) and (3.6) leads to

$$\|Qg\|_{p,v}\geqslant (1-r_0)\|g\|_{p,v} \quad \text{for all } g\in V_v. \tag{3.7}$$

On the other hand, it follows from (2.3), (2.11) and (3.4) that for any $g\in V_v$,

$$\begin{aligned} \|Qg\|_{p,v} &= \left\| \sum_{\gamma\in\Gamma}\langle g,\psi_\gamma\rangle u_\gamma(\cdot)v(\cdot) \right\|_p \\ &\leqslant \Big(\sup_{z\in[-\delta_0/2,\delta_0/2]^d} w(z) \Big) \left\| \sum_{\gamma\in\Gamma} \big|\langle g,\psi_\gamma\rangle\big| v(\gamma)u_\gamma(\cdot) \right\|_p \\ &\leqslant \Big(\sup_{z\in[-\delta_0/2,\delta_0/2]} w(z) \Big) \Big(\sup_{\gamma\in\Gamma}\|u_\gamma\|_1 \Big)^{1/p} \big\| \big(\langle g,\psi_\gamma\rangle\big)_{\gamma\in\Gamma} \big\|_{\ell^p_v} \\ &\leqslant \delta_0^{d/p}\big(A_\Gamma(\delta_0)\big)^{-1/p} \Big(\sup_{z\in[-\delta_0/2,\delta_0/2]} w(z) \Big) \big\| \big(\langle g,\psi_\gamma\rangle\big)_{\gamma\in\Gamma} \big\|_{\ell^p_v}. \end{aligned} \tag{3.8}$$

Then the lower bound estimate of $\|(\langle g,\psi_\gamma\rangle)_{\gamma\in\Gamma}\|_{\ell^p_v}$ in (3.3) follows from (3.7) and (3.8).

For $p=\infty$ and $g\in V_v$,

$$\begin{aligned} \big\| \big(\langle g,\psi_\gamma\rangle\big)_{\gamma\in\Gamma} \big\|_{\ell^\infty_v} &= \sup_{\gamma\in\Gamma}\big|\langle g,\psi_\gamma\rangle\big| v(\gamma) \\ &\leqslant \sup_{\gamma\in\Gamma} v(\gamma) \int_{\mathbb{R}^d} \big(\big|g(y)-g(\gamma)\big| + \big|g(\gamma)\big| \big) \big|\psi_\gamma(y)\big|\,dy \\ &\leqslant M \sup_{\gamma\in\Gamma} \big(\omega_{a/2}(g)(\gamma) + \big|g(\gamma)\big| \big) v(\gamma) \end{aligned}$$

$$\leqslant M\Big(1+\sup_{x\in\mathbb{R}^d}\big\|\omega_{a/2}(K)(x,x-\cdot)\big\|_{1,w}\Big)\|g\|_{\infty,v}$$
$$=(M+r_0)\|g\|_{\infty,v}, \tag{3.9}$$

where we have used (2.11) and Proposition 2.1 to obtain the last inequality. Similarly for $1\leqslant p<\infty$ and $g\in V_v$,

$$\begin{aligned}
\big\|\big(\langle g,\psi_\gamma\rangle\big)_{\gamma\in\Gamma}\big\|_{\ell^p_v} &= \Bigg(\sum_{\gamma\in\Gamma}\bigg|\int_{\mathbb{R}^d} g(y)\psi_\gamma(y)\,dy\bigg|^p\big(v(\gamma)\big)^p\Bigg)^{1/p}\\
&\leqslant M^{(p-1)/p}\Bigg(\sum_{\gamma\in\Gamma}\big(v(\gamma)\big)^p\int_{\mathbb{R}^d}\big|g(y)\big|^p\big|\psi_\gamma(y)\big|\,dy\Bigg)^{1/p}\\
&\leqslant M^{(p-1)/p}\big(\delta_0^{-d}B_\Gamma(\delta_0)\big)^{1/p}\Bigg(\sum_{\gamma\in\Gamma}\big(v(\gamma)\big)^p\int_{x,y\in\mathbb{R}^d}\big|g(y)\big|^p\big|\psi_\gamma(y)\big|u_\gamma(x)\,dy\,dx\Bigg)^{1/p}\\
&\leqslant M^{(p-1)/p}\big(\delta_0^{-d}B_\Gamma(\delta_0)\big)^{1/p}\\
&\quad\times\Bigg(\sum_{\gamma\in\Gamma}\big(v(\gamma)\big)^p\int_{x,y\in\mathbb{R}^d}\big(\big|g(x)\big|+\omega_{(\delta_0+a)/2}(g)(x)\big)^p\big|\psi_\gamma(y)\big|u_\gamma(x)\,dy\,dx\Bigg)^{1/p}\\
&\leqslant M\big(\delta_0^{-d}B_\Gamma(\delta_0)\big)^{1/p}\Bigg(\sum_{\gamma\in\Gamma}\big(v(\gamma)\big)^p\int_{x\in\mathbb{R}^d}\big(\big|g(x)\big|+\omega_{(\delta_0+a)/2}(g)(x)\big)^p u_\gamma(x)\,dx\Bigg)^{1/p}\\
&\leqslant M\big(\delta_0^{-d}B_\Gamma(\delta_0)\big)^{1/p}\Big(\sup_{z\in[-\delta_0,\delta_0/2]^d} w(z)\Big)\\
&\quad\times\Bigg(\sum_{\gamma\in\Gamma}\int_{x\in\mathbb{R}^d}\big(\big|g(x)\big|+\omega_{(\delta_0+a)/2}(g)(x)\big)^p\big(v(x)\big)^p u_\gamma(x)\,dx\Bigg)^{1/p}\\
&\leqslant M\Big(\sup_{z\in[-\delta_0/2,\delta_0/2]^d} w(z)\Big)\big(\delta_0^{-d}B_\Gamma(\delta_0)\big)^{1/p}\big(\|g\|_{p,v}+\big\|\omega_{(\delta_0+a)/2}(g)\big\|_{p,v}\big)\\
&\leqslant\Big(\sup_{z\in[-\delta_0/2,\delta_0/2]^d} w(z)\Big)\big(\delta_0^{-d}B_\Gamma(\delta_0)\big)^{1/p}(M+r_0)\|g\|_{p,v},
\end{aligned}\tag{3.10}$$

where the third inequality follows from the supporting properties for the functions ψ_γ and u_γ for each $\gamma\in\Gamma$, the fifth inequality holds by (2.3) and (3.4), and the last inequality is obtained from (2.11) and Proposition 2.1. Therefore the upper bound estimate of $\|(\langle g,\psi_\gamma\rangle)_{\gamma\in\Gamma}\|_{\ell^p_v}$ in (3.3) follows from (3.9) and (3.10). $\square$

Remark 3.2. We can give one example of bounded uniform partition of unity (BUPU) associated with the covering $\{\gamma+[-\delta_0/2,\delta_0/2]^d\}_{\gamma\in\Gamma}$ of $\mathbb{R}^d$ by the following

$$u_\gamma(x)=\frac{\chi_{\gamma+[-\delta_0/2,\delta_0/2]^d}(x)}{\sum_{\gamma'\in\Gamma}\chi_{\gamma'+[-\delta_0/2,\delta_0/2]^d}(x)},\quad \gamma\in\Gamma. \tag{3.11}$$

4. Iterative reconstruction algorithm

In this section, we develop an iterative algorithm to recover a signal f in the weighted reproducing kernel subspace V_v of L^p_v from its weighted samples $\langle f,\psi_\gamma\rangle$, $\gamma\in\Gamma$, taken a relatively-separated subset Γ of $\mathbb{R}^d$ with sufficient small gap.

Theorem 4.1. *Let $1\leqslant p\leqslant\infty$, $0<\delta_0,a<\infty$, w be a continuous submultiplicative weight, v be a continuous w-moderate weight, K be a measurable function on $\mathbb{R}^d\times\mathbb{R}^d$ satisfying the decay condition* (2.5) *and the regularity condition* (2.18) *for $\delta=(\delta_0+a)/2$, T be the integral idempotent operator in* (2.6) *with kernel K, r_0 be defined as in* (3.2)*, V_v be the reproducing kernel subspace of L^p_v associated with the operator T, Γ be a relatively-separated subset of $\mathbb{R}^d$ with δ_0 being its gap, $U=\{u_\gamma\}_{\gamma\in\Gamma}$ is a bounded uniform partition of unity (i.e.,* (3.4) *holds), and $\Psi=\{\psi_\gamma\}_{\gamma\in\Gamma}$ be a family of weighted sampling functionals with $M:=\sup_{\gamma\in\Gamma}\|\psi\|_1<\infty$, $\int_{\mathbb{R}^d}\psi_\gamma(x)\,dx=1$ for each $\gamma\in\Gamma$ and ψ_γ is supported in $\gamma+[-a/2,a/2]^d$ for each $\gamma\in\Gamma$. If* (3.2) *holds, then given the weighted samples $y(\gamma)=\langle g,\psi_\gamma\rangle$, $\gamma\in\Gamma$, of a signal $g\in V_v$, the following* iterative reconstruction algorithm

$$\begin{cases} g_0 = \sum_{\gamma\in\Gamma} g(\gamma)Tu_\gamma, \\ g_n = g_0 + g_{n-1} - \sum_{\gamma\in\Gamma} \langle g_{n-1}, \psi_\gamma\rangle Tu_\gamma, \quad n \geqslant 1, \end{cases} \tag{4.1}$$

converges to g exponentially. Precisely,

$$\|g_n - g\|_{p,v} \leqslant \frac{\|T\|r_0^{n+1}}{1-r_0}\|g_0\|_{p,v} \quad n \geqslant 1, \tag{4.2}$$

where $\|T\|$ is the operator norm on L^p_v.

Remark 4.1. The exponential convergence (4.2) of the iterative algorithm (4.1) is given in [12] for the ideal sampling and trivial weight case, i.e., $\langle g, \psi_\gamma\rangle = g(\gamma)$ for all $\gamma \in \Gamma$, and $v = w \equiv 1$. The iterative reconstruction algorithm (4.1) is known as the *iterative approximation-projection reconstruction algorithm*, which was introduced in [10] for reconstructing band-limited signals, and was later generalized to signals in shift-invariant spaces [1], and in reproducing kernel subspaces of L^p [12].

Proof of Theorem 4.1. Let Q be the linear operator defined in (3.5). Then we can reformulate the iterative reconstruction algorithm (4.1) as

$$g_0 = TQg \quad \text{and} \quad g_n = g_0 + g_{n-1} - TQg_{n-1} \quad \text{for } n \geqslant 1. \tag{4.3}$$

Therefore $g_{n+1} - g_n = (T - TQ)(g_n - g_{n-1})$ for all $n \geqslant 1$. Applying the above equation iteratively and using $T^2 = T$ yields

$$\begin{aligned} g_n - g_{n-1} &= (T-TQ)^{n-1}(g_1 - g_0) \\ &= (T-TQ)^n TQg = T(T-QT)^n TQg, \quad n \geqslant 1. \end{aligned} \tag{4.4}$$

By (3.6), $\sum_{k=1}^\infty (T-QT)^k$ is well defined and satisfies

$$T = \left(\sum_{k=1}^\infty (T-QT)^k\right)QT + QT. \tag{4.5}$$

By (2.11) and (4.4),

$$g_n = T\left(\sum_{k=1}^n (T-QT)^k QT + QT\right)g, \quad n \geqslant 1. \tag{4.6}$$

Combining (3.6), (4.3), (4.4), (4.5) and (4.6) proves exponential convergence of the sequence $\{g_n\}_{n\geqslant 1}$ and also (4.2) because

$$\begin{aligned} \|g_n - g\|_{p,v} &\leqslant \sum_{k=n}^\infty \|g_{k+1} - g_k\|_{p,v} \leqslant \|T\| \sum_{k=n}^\infty \left\|(T-QT)^{k+1}g_0\right\|_{p,v} \\ &\leqslant \|T\|\|g_0\|_{p,v} \sum_{k=n}^\infty r_0^{k+1} = \|T\|\|g_0\|_{p,v} r_0^{n+1}/(1-r_0). \qquad \square \end{aligned}$$

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